SARVALOKUM: LAW AND SOCIETY-
MULTIDISCIPLINARY NATIONAL PEER-REVIEWED JOURNAL
VOLUME-3, ISSUE: 1, MAY 2026
ISSN: 3049-0006 (ONLINE)

A Hybrid DTM-FDM Framework for Solving Linear & Nonlinear PDEs via Similarity Transformations

***Brijal M. Parmar*[1]*, Shama M. Mulla*[2] *& Vahisht K. Tamboli*[3]**

[1]*Ph.D. Research Scholar, Department of Mathematics, Sarvajanik College of Engineering and Technology, Surat, Gujarat, 395001 INDIA*

[2]*Associate Professor, Department of Mathematics, Sarvajanik College of Engineering and Technology, Surat, Gujarat, 395001 INDIA,*

[3]*Ph.D. Research Scholar, Department of Mathematics, Veer Narmad South Gujarat University, Surat, Gujarat, 395001 INDIA*

Abstract

This study investigates the use of Lie group transformations, particularly translation transformations, to reduce complex partial differential equations (PDEs) into ordinary differential equations (ODEs) for obtaining similarity solutions. The linear and nonlinear ODEs derived through similarity transformations are solved using two methods: the semi-analytical Differential Transform Method (DTM) and the numerical Finite Difference Method (FDM). This study introduces a hybrid framework that integrates Lie group symmetry transformations with DTM and FDM to overcome the limitations of traditional methods. DTM, based on Taylor series expansion, properly handles nonlinear terms through convolution properties and provides exact power-series solutions with rapid convergence for linear ODEs, while FDM delivers robust and accurate numerical approximations, especially for nonlinear systems. This integration significantly reduces complexity and improves accuracy in solving nonlinear differential equations. Finally, we conclude that this novel hybrid approach, combining Lie group symmetry transformations with DTM and FDM, obtains accurate and efficient solutions to nonlinear ODEs, offering important advancements in mathematical physics, computational science, and engineering applications such as mechanics, fluid dynamics, and heat transfer.

Keywords: Lie Group Transformations, Differential Transform Method (DTM), Finite Difference Method (FDM), Translation Transformations, Nonlinear PDEs.

Mathematical Subject Classification:

Table 4: Nomenclature - Symbols and their meanings.

Symbol	Define
x, t	Independent variables: spatial coordinate and time
$u(x,t)$	Dependent variable (e.g., displacement in wave equation)
c	Wave speed (constant)
η	Similarity variable (often $\eta = x \pm ct$)
$f(\eta)$	Group-invariant (similarity) solution function
ξ	Transformation parameter in Lie group
ε	Infinitesimal parameter in Lie group transformations
$\phi(x,t)$	Arbitrary function in general solution
$U(x,y)$	Velocity component in boundary layer equation

α	Thermal diffusivity (in heat equation)
k	Nonlinear coefficient (in nonlinear heat equation)

1. Introduction

In many undergraduate applied mathematics courses, students learn to solve classic problems like the wave equation, heat equation, and boundary layer equations using problem-specific techniques (Smith, 1985). While suitable for particular cases, such techniques possess restricted generality in the context of nonlinear partial differential equations (PDEs). Typically, the solution techniques presented are specific to the problem under consideration, showing limited applicability to general nonlinear PDEs. Despite this, general methodologies for deriving exact solutions of linear and nonlinear PDEs exist and may be incorporated into academic courses such as differential equations (DEs) and mechanics (Bluman & Kumei, 1989a; Olver, 1986). Among these general methods, Lie group transformations and their special forms such as scaling, translation, and spiral transformations prove to be highly effective tools (Ibragimov, 1994). Such transformations serve two primary purposes: (1) they can generate new solutions based on existing ones, and (2) they enable the derivation of similarity (group-invariant) solutions (Olver, 1986). Similarity transformations are particularly valuable because they reduce systems of PDEs with multiple independent variables to systems with fewer independent variables, and in the best cases, to ordinary differential equation (ODE) (Ibragimov, 1994). Specifically, the translation transformation applied to the linear wave equation yields d'Alembert type traveling wave solutions in both directions, eliminating the need to combine it with other transformations in the standard derivation process (Olver, 1986; Zaitsev & Polyanin, 2002). However, in more complex cases, translation is often combined with scaling or other symmetries for nontrivial reductions. Beyond the linear wave equation, this study also examines the Viscous Burgers' equation (Abd-el-Malek & El-Mansi, 2000), a fundamental nonlinear PDEs that arises in multiple areas of applied mathematics, including fluid mechanics, nonlinear acoustics, and gas dynamics. Unlike the linear case, the Burgers' equation contains the nonlinear convective term $\left(uu_x\right)$ and the dissipative term $\left(u_{xx}\right)$, making it an ideal candidate to test the effectiveness of the proposed hybrid DTM-FDM framework. Although numerous analytical and numerical techniques exist in contemporary mathematical modelling and engineering applications, conventional methods for solving nonlinear PDEs remain constrained by significant limitations. These limitations include high computational complexity, slow convergence rates, and reduced accuracy, particularly when dealing with nonlinear terms (Rashidi, 2010; Smith, 1985). Specifically, relying on a single method is frequently inadequate for effectively managing the nonlinear terms essential in complex mathematical models. This challenge requires the development of hybrid frameworks that combine the strengths of multiple solution techniques to overcome these limitations. This study explores the application of Lie group transformations, with a particular focus on translation transformations closely associated with translation invariance to reduce selected PDEs into ordinary differential equations (ODEs) for the purpose of deriving similarity and traveling wave solutions. The resulting linear and nonlinear ODEs are then solved using two complementary approaches: the semi-analytical Differential Transform Method (DTM) and the numerical Finite Difference Method (FDM) (Rashidi, 2010; Zhou, 1986).

The DTM, created on Taylor series expansion, is a robust semi-analytical technique capable of proficiently handling nonlinear terms via its convolution properties, while offering exact power series solutions with rapid convergence for linear ODEs (Zhou, 1986). On the other hand, FDM offers robust and accurate numerical approximations, especially for nonlinear systems (Rashidi, 2010). By combining these two complementary techniques, the proposed framework significantly enhances solution precision and overcomes the inherent limitations of conventional methods for nonlinear DEs.

This work proposes a novel hybrid framework that integrates Lie group symmetry transformations with DTM and FDM. The proposed hybrid approach offers several significant advantages: (1) a substantial reduction in computational complexity; (2) effective treatment of nonlinear terms through the convolution property of DTM; (3) improved accuracy and numerical stability due to the robustness of FDM; and (4) wide applicability to practical problems in mechanics, fluid dynamics, and heat transfer.

The objectives of this research are as follows:

1. To review and simplify similarity methods for solving PDEs that can be integrated into undergraduate mathematics courses, making these advanced techniques more accessible to students.
2. To demonstrate the application of special forms of Lie group transformations specifically scaling, translation, and spiral group transformations as practical alternatives to general Lie group methods for finding exact solutions of linear and nonlinear DEs.
3. To apply translation transformation methods to fundamental equations in mathematical physics, particularly the wave equation in mechanics and boundary layer equations in fluid mechanics, to derive similarity solutions.
4. To solve nonlinear ODEs resulting from similarity transformations using both analytical and numerical approaches, specifically the DTM and the FDM.
5. To establish and demonstrate the properties, transformation rules, and recursive relations of the DTM for solving DEs effectively.
6. To validate and compare the accuracy of analytical (DTM) and numerical (FDM) solutions through computational examples and graphical representations.
7. To derive the well-known moving wave solution of the nondimensional wave equation using translation transformation, demonstrating waves traveling in both right and left directions.

The rest of this paper is organized as follows: Section 2 presents a comprehensive literature review; Section 3 outlines the mathematical formulation of the wave equation and presents the applied translation transformation; Section 4 demonstrates the analytical solution via DTM; Section 5 presents the numerical solution via FDM; Section 6 provides results and discussion; and Section 7 concludes the study with key findings and future research directions.

2. Literature Review

The application of symmetry methods, particularly Lie group transformations, has long served as a cornerstone in the analytical treatment of DEs within mathematical physics and engineering disciplines (Bluman & Kumei, 1989a; Hydon, 2000). In undergraduate applied mathematics education, the focus is primarily on teaching solution approaches that are developed for individual problems, with particular attention given to classical equations such as the heat equation, wave equation, and boundary layer equations. While these methods prove effective for particular cases, they exhibit limited applicability to general linear and nonlinear PDEs. Despite this, general frameworks for deriving exact solutions to linear and nonlinear PDEs exist and may be incorporated into wider curricula such as DEs and mechanics courses (Bluman & Kumei, 1989b; Olver, 1986).

Lie group theory provides a systematic and powerful framework for deriving exact solutions of DEs (Stephani, 1990). Lie group transformations, together with their particular cases such as scaling, translation, and spiral transformations, serve two fundamental purposes: (1) they allow the construction of new solutions from existing ones, and (2) they facilitate the determination of group-invariant similarity solutions (Ibragimov, 1994; Olver, 1986). Similarity transformations are especially powerful because they systematically reduce PDEs with multiple independent variables to systems with fewer independent variables, and in optimal cases, to ODEs (Dresner, 1983). These transformation techniques have been widely utilized to study fundamental equations in mathematical physics, including the linear wave equation associated with mechanics and the boundary layer equation encountered in fluid mechanics (Zaitsev & Polyanin, 2002). Translation transformations, when applied alone, often yield trivial solutions. However, when combined with other symmetries, they produce significant results, such as the well-known d'Alembert type moving wave solutions of the linear wave equation (Courant & Hilbert, 1989). Among various transformation techniques, scaling transformations have demonstrated significant capability in yielding nontrivial similarity solutions for a wide spectrum of equations (Dresner, 1983).

For solving the resulting ODEs (linear or nonlinear) obtained from similarity reductions, various analytical and numerical methods have been extensively developed over the past few decades. The DTM, first proposed by Zhou in 1986 in the context of electrical circuit analysis, has developed into a robust semi-analytical approach founded on Taylor series expansion (Zhou, 1986). DTM handles nonlinear terms effectively through its convolution properties and provides exact power-series solutions with rapid convergence, particularly for linear ODEs (Ayaz, 2003; Erfani et al., 2010). This method converts DEs into corresponding algebraic forms, thereby greatly reducing the complexity of the solution procedure. As a result of these benefits, DTM has been extensively employed to address a wide range of nonlinear problems arising in fluid dynamics, heat transfer, mechanics, and nonlinear oscillatory systems (Arikoglu & Ozkol, 2005). The applicability of the DTM to nonlinear fractional DEs was demonstrated by Odibat and Momani (2008) (Odibat & Momani, 2008), while Hassan (2004) (Abdel-Halim Hassan, 2004) successfully employed the technique to address higher order boundary value problems. Further developments were reported by Rashidi et al. (Erfani et al., 2010), who extended the method to couple nonlinear fluid mechanics systems and obtained results in close agreement with exact solutions.

The FDM remains one of the most robust and widely used numerical approaches for approximating solutions of DEs (Smith, 1985). FDM discretizes the computational domain and replaces derivatives with finite difference approximations using forward, backward, or central difference schemes (LeVeque, 2007). The FDM is well known for providing stable and accurate numerical solutions, especially in the treatment of nonlinear PDEs and boundary value problems. Nevertheless, achieving higher accuracy through refined spatial discretization leads to a substantial increase in computational cost. Despite this limitation, FDM continues to be preferred in engineering applications due to its simplicity, flexibility, and well-established theoretical foundation. Strikwerda (Strikwerda, 2004) provided comprehensive theoretical analysis of finite difference schemes for PDEs, while LeVeque (LeVeque, 2007) presented advanced finite difference methods for both ODEs and PDEs with emphasis on stability and convergence analysis.

Recent studies have increasingly emphasized hybrid strategies that couple analytical methods with numerical schemes to simultaneously improve accuracy and lower computational complexity (Afrin et al., 2011; Moosavi Noori & Taghizadeh, 2020). The underlying motivation of hybrid methods is to exploit the complementary advantages of analytical and numerical techniques: analytical approaches yield deeper insight into solution structures and exact representations, whereas numerical methods provide robustness and adaptability when dealing with complex problems. In this context, several studies have combined Lie group symmetry analysis with either the DTM or the FDM on an individual basis. Biazar and Eslami (Zhou, 1986) applied DTM combined with symmetry analysis to fractional gas dynamics equations, demonstrating improved computational efficiency. Kumar et al. (Kumar et al., 2022) developed a hybrid approach using Lie symmetry with DTM for nonlinear PDEs and reported enhanced accuracy compared to standalone methods.

Despite notable progress, a thoroughly integrated hybrid DTM-FDM approach, particularly one that utilizes translation transformations to reduce PDEs into ODEs, remains scarcely explored in current research. Most reported studies treat DTM and FDM separately or merge them without the systematic application of Lie group transformations as a preprocessing stage. Furthermore, while translation transformations have been recognized as valuable tools in symmetry analysis, their systematic application combined with hybrid numerical-analytical approaches has not been adequately explored (Zaitsev & Polyanin, 2002). Combining translation transformation driven PDE reduction with a hybrid DTM-FDM solution framework presents a substantial potential for enhancing computational methodologies applied to nonlinear DEs. The reviewed literature indicates that although individual methods Lie group transformations, DTM, and FDM possess distinct strengths, an integrated approach that systematically combines all three techniques is essential for efficient and accurate solution of nonlinear PDEs. Despite their individual strengths, traditional methods often face limitations when dealing with highly nonlinear PDEs, including computational complexity, slow convergence, and restricted accuracy (Smith, 1985).

In order to overcome the identified research gap, this study applies Lie group translation transformations for reducing complex PDEs to ODEs, followed by their solution using an integrated hybrid DTM-FDM approach. This integrated approach aims to reduce computational complexity, enhance solution accuracy, overcome convergence limitations, and provide efficient solutions for nonlinear DEs arising in applied mathematics and engineering applications. The proposed framework presents multiple potential benefits: it enables systematic reduction of PDEs using translation symmetries, provides rapid semi-analytical solutions via DTM for smooth problems, delivers robust numerical approximations through FDM for complex nonlinear systems, and enhances overall computational efficiency by strategically integrating and selecting the appropriate methods.

3. Mathematical Formulation

This study uses similarity methods for the solution of PDEs. Initially, fundamental model equations, including the wave equation and the heat equation, are considered to illustrate the proposed methodology. Using similarity transformations such as scaling, translation, and spiral group transformations, the governing PDEs are converted into ODEs, which are comparatively easier to solve. These transformations represent special forms of general Lie group transformations and are applicable to a wide class of problems.

Governing Equations

In mathematical physics, similarity methods provide powerful tools for solving PDEs and ODEs. The translation transformation method represents one of the fundamental similarity transformations, alongside scaling and spiral group transformations.

The governing equation for wave propagation in one spatial dimension is the classical linear wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ denotes the transverse displacement (or wave function) at spatial position x and time t, and $c>0$ is the constant wave propagation speed.

In many normalized or simplified models (e.g., unit wave speed $c=1$, common in numerical studies of vibrating strings or acoustic waves), the equation reduces to, for a single spatial co-ordinate, the linear wave equation is,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where, $u(x,t)$ is the wave displacement, x is the spatial coordinate, t is the time variable.

Without loss of generality, we set $c=1$ for simplicity in the following derivation.

Equation: (1) Linear Wave Equation

The one-dimensional linear wave equation governing the transverse displacement $u(x,t)$ of a vibrating string (or wave-propagating medium) along a single spatial coordinate x at time t, where $c>0$ is the constant wave propagation speed. This equation states that the acceleration of the string element is proportional to its curvature, ensuring wave propagation at constant speed c.

Translation Transformation

The translation transformation for (1) is

$$x^* = x + \varepsilon a,\ t^* = t + \varepsilon b,\ u^* = u + \varepsilon c \tag{2}$$

where ε is the transformation parameter and a, b, and c are to be determined from the equation.

Substituting (2) into (1), we see that no restrictions are imposed on the parameters a, b and c .For the linear homogeneous wave equation, the shift in u is trivial, hence we set $c = 0$ to obtain non-trivial reductions.

Now, equation (2) implies

$$x^* = x + \varepsilon a$$
$$\Rightarrow x^* - x = \varepsilon a$$
$$\Rightarrow dx = x^* - x = \varepsilon a \qquad \left[\text{Assuming}\, dx = x^* - x\right]$$
$$\Rightarrow dx = \varepsilon a$$
$$\Rightarrow \frac{dx}{a} = \varepsilon$$

$$\Rightarrow \boxed{\frac{dx}{a} = \varepsilon} \tag{3}$$

- Again, equation (2) implies

$$t^* = t + \varepsilon b$$
$$\Rightarrow t^* - t = \varepsilon b$$
$$\Rightarrow dt = t^* - t = \varepsilon b \qquad \left[\text{Assuming}\, dt = t^* - t\right]$$
$$\Rightarrow dt = \varepsilon b$$
$$\Rightarrow \frac{dt}{b} = \varepsilon$$

$$\Rightarrow \boxed{\frac{dt}{b} = \varepsilon} \tag{4}$$

- Again, equation (2) implies

$$u^* = u + \varepsilon c$$
$$\Rightarrow u^* - u = \varepsilon c$$
$$\Rightarrow du = u^* - u = \varepsilon c \qquad \left[\text{Assuming}\, du = u^* - u\right]$$
$$\Rightarrow du = \varepsilon c$$
$$\Rightarrow \frac{du}{c} = \varepsilon$$

$$\Rightarrow \boxed{\frac{du}{c} = \varepsilon} \tag{5}$$

- From equations (3), (4), and (5), we get

$$\therefore \frac{dx}{a} = \frac{dt}{b} = \frac{du}{c} = \varepsilon$$

- Therefore, the equation allows arbitrary translations in all co-ordinates. For transformation (2), the equivalent differential system is

$$\frac{dx}{a} = \frac{dt}{b} = \frac{du}{c} \tag{6}$$

- Choosing $c = 0$ and $a = mb$, and solving (6) by the method of characteristics, we have

$$\frac{dx}{a}=\frac{dt}{b}=\frac{du}{c}$$

$$\Rightarrow \frac{dx}{mb}=\frac{dt}{b}=\frac{du}{0} \qquad (7)$$

$$\Rightarrow \boxed{u=F(\eta)} \quad \left[\text{where } F(\eta) \text{ is integral constant}\right] \qquad (8)$$

- Again, equation (7) implies

$$\frac{dx}{mb}=\frac{dt}{b}$$

$$\Rightarrow dx = m\,dt$$

- Taking integration on both sides of above equation, we get

$$\Rightarrow \int dx = m\int dt \quad [\because \text{As integral operator is linear}]$$

$$\Rightarrow x = mt+\eta \quad \left[\text{where } \eta \text{ is integral constant}\right]$$

$$\Rightarrow x-mt=\eta$$

$$\Rightarrow \boxed{\eta = x-mt} \qquad (9)$$

- Rewriting equation (1),

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left[F(\eta)\right]=\frac{\partial^2}{\partial x^2}\left[F(\eta)\right] \qquad \text{[By eq.(8)]}$$

$$\Rightarrow \frac{\partial}{\partial t}\left[F'(\eta)(-m)\right]=\frac{\partial}{\partial x}\left[F'(\eta)\cdot 1\right]$$

$$\Rightarrow (-m)F''(\eta)(-m)=F''(\eta)\cdot 1$$

$$\Rightarrow m^2F''(\eta)=F''(\eta)$$

$$\Rightarrow m^2F''(\eta)-F''(\eta)=0$$

$$\Rightarrow \left[m^2-1\right]F''(\eta)=0$$

$$\Rightarrow \boxed{\left(m^2-1\right)F''=0} \qquad (10)$$

- We now have two choices; either $F''=0$ or $m=\pm 1$.
- Let us take

$$F''(\eta)=0$$

- Taking integration on both sides of above equation, we get

$F'(\eta)=c_1$; where c_1 is integrating constant

- Taking integration on both sides of above equation, we get

$F(\eta)=c_1\eta+c_2$; where c_1 & c_2 are integrating constants

$$\Rightarrow \boxed{F(\eta)=c_1\eta+c_2} \tag{11}$$

- Replacing the value of $\eta = x - mt$ in above equation, we get

$$F(x-mt)=c_1(x-mt)+c_2$$
$$\Rightarrow F(x-mt)=c_1x-c_1mt+c_2$$
$$\Rightarrow F(x-mt)=(c_1)x+(-c_1m)t+c_2$$
$$\Rightarrow F(x-mt)=k_1x+k_2t+k_3$$

; where $k_1=c_1,\ k_2=(-c_1m),\ k_3=c_2$

$$\Rightarrow F(\eta)=k_1x+k_2t+k_3$$

$$\Rightarrow u=k_1x+k_2t+k_3 \qquad \text{[By eq.(8)]}$$

$$\Rightarrow \boxed{u=k_1x+k_2t+k_3} \tag{12}$$

- The first choice $F''=0$ leads to the following equations (11), and (12) as follows:

$\boxed{F=c_1\eta+c_2}$ and

$\boxed{u=k_1x+k_2t+k_3}$

- which is the trivial solution that can be found directly by examining eq.(1).
- For the second choice $m=\pm 1$, we obtain the well-known solution

$$\boxed{u=F_1(x-t)+F_2(x+t)} \tag{13}$$

- where F_1 represents the wave travelling to the right and F_2 represents the wave travelling to the left.
- We didn't consider any specific boundary conditions for the analysis and hence, the solutions appear in their general form.
- Equation (10) is linear ordinary differential equation (ODE).

In simple terms, we converted highly complex PDEs into a system of linear ODEs, which can then be solved using DTM and FDM methods.

Let us solve equation (10) which are the system of linear ODEs by both analytical and numerical methods.

Viscous Burgers' Equation:

The standard form of the nonlinear viscous Burgers' equation is given by:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=v\frac{\partial^2 u}{\partial x^2} \tag{14}$$

Where: $u(x,t)$ is the velocity field, v is the kinematic viscosity parameter.

Applying Translation Transformation: We introduce the similarity variable $\xi=x-ct$, where c is the wave speed. The derivatives are transformed as:

$$\frac{\partial u}{\partial t}=-c\frac{\partial \phi}{\partial \xi},\ \frac{\partial u}{\partial x}=\frac{\partial \phi}{\partial \xi},\ \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 \phi}{\partial \xi^2}$$

Substituting these into the governing equation (14):

$$-c\phi'+\phi\phi'=v\phi''$$

Integrating once with respect to ξ :

$$-c\phi+\frac{1}{2}\phi^2=v\phi'+K$$

For simplicity, we can assume the integration constant $K=0$ for boundary conditions where; $\phi\to 0$ as $\xi\to\infty$.

Rearranging, we obtain the First-Order Nonlinear ODE:

$$\phi'=\frac{1}{v}\left(\frac{\phi^2}{2}-c\phi\right) \tag{15}$$

4. Differential Transform Method (DTM)

The DTM, originally introduced by Zhou (1986) for solving linear and nonlinear problems in electrical circuits, is a powerful semi-analytical technique based on the Taylor series expansion. DTM transforms DEs into a system of algebraic equations through recursive relations, making it particularly effective for handling both linear and nonlinear terms. The method provides power-series solutions with rapid convergence, especially for linear ODEs. DTM is a semi-analytical technique that is used for the solution of ODEs and PDEs. This method is based on Taylor series expansion, and is simple for handling nonlinear terms.

Basic Principles of DTM: -

If the function $f(\xi)$ be an analytic function in the domain of interest. The differential transform of $f(\xi)$ at a point $\xi=\xi_0$ (usually $\xi=0$) is defined as:

$$F(k)=\frac{1}{k!}\left[\frac{d^k}{d\xi^k}f(\xi)\right]_{\xi=\xi_0}$$

where U(k) is the transformed function (spectrum) at ξ_0.

The inverse differential transform is given by:

$$f(\xi)=\sum_{k=0}^{\infty}F(k)(\xi-\xi_0)^k$$

For practical computations, the series is truncated at a finite number of terms to obtain an approximate solution.

Fundamental Properties and Transformation Rules: -

Table 5 : Rules for the DTM- Mapping from original to transformed forms.

Original form	Transformed form

$u(\xi)$	$U(k)$
$u'(\xi)$	$(k+1)U(k+1)$
$u''(\xi)$	$(k+1)(k+2)U(k+2)$

Application of DTM to the Reduced ODE from the Wave Equation:

We will apply DTM to the system of linear ODEs term by term, using transformation rules for derivatives terms as mentioned in above Table.

Now, applying DTM transformations of equation (10), we get

$$\left(m^2-1\right)F''\xrightarrow{\text{DTM}}\left(m^2-1\right)(K+2)(K+1)F(k+2)$$

$$\boxed{\left(m^2-1\right)(K+2)(K+1)F(k+2)=0} \tag{16}$$

There are two cases:

Case: 1 $m^2-1=0\Rightarrow m=\pm1$

Substituting $m=\pm1$ in equation (16),

$$\left[(\pm1)^2-1\right](k+2)(k+1)F(k+2)=0$$

$$\Rightarrow[1-1](k+2)(k+1)F(k+2)=0$$

$$\Rightarrow\boxed{0=0}$$

Equation is always true, meaning there are no restrictions on $F(k+2)$.

Case: 2 $m^2-1\neq0$

$$F(k+2)=\frac{0}{\left(m^2-1\right)(K+2)(K+1)};m^2-1\neq0$$

$$\Rightarrow\boxed{F(k+2)=0 \quad k\in0,1,2,3.....} \tag{17}$$

Substituting $k=0$ in equation (17),

$$\Rightarrow F(0+2)=0$$

$$\Rightarrow\boxed{F(2)=0}$$

Substituting $k=1$ in equation (17),

$$\Rightarrow F(1+2)=0$$

$$\Rightarrow\boxed{F(3)=0}$$

Substituting $k=2$ in equation (17),

$$\Rightarrow F(2+2)=0$$

$$\Rightarrow\boxed{F(4)=0}$$

$$\boxed{F(2)=F(3)=F(4)=F(5)=......=0} \tag{18}$$

The DTM recursive relation shows that all higher-order terms are zero.

i.e. $F(2)=F(3)=F(4)=F(5)=........=0$. This means that the solution is limited to only a first-degree polynomial. Only $F(0)$ & $F(1)$ terms remain. The general power series expansion for $F(\eta)$ in DTM is:

$$F(\eta)=\sum_{k=0}^{\infty}F(k)\eta^k$$

The recursive formula $F(k+2)=0$ tells us that every term with index $k+2$ is zero for all k .

Now, explain $F(\eta)$

$$F(\eta)=F(0)\eta^0+F(1)\eta^1+F(2)\eta^2+F(3)\eta^3+.......$$

$$F(\eta)=F(0)\cdot 1+F(1)\eta+0\cdot\eta^2+0\cdot\eta^3+....... \text{ [By equation (18)]}$$

$$\boxed{F(\eta)=F(0)+F(1)\eta}$$

which is a linear function.

We now see that DTM restricts the solution to be a linear function of η . This matches the classical general solution of a first-order linear ODE:

$$F(\eta)=F(0)+F(1)\eta$$

$$\boxed{F(\eta)=c_1\eta+c_2}$$

were, $\boxed{c_1=F(1),\, c_2=F(0)}$

Since all higher-order terms vanish, the solution is only a first-degree polynomial. This matches the expected analytical solution for simple linear ODEs. This explains why DTM is very effective for solving linear differential equations. Using DTM, we transformed the given ODE and obtained a linear function as the solution because the recursive formula eliminated all higher-order terms. This is the exact general solution of the linear ODE $f''(\eta)=0$, demonstrating that DTM provides the exact analytical solution in finite terms for linear problems.

Applying the DTM to the nonlinear ODE:

Now, equation (15), implies

$$\phi'(\xi)=\frac{1}{2v}\phi^2(\xi)-\frac{c}{v}\phi(\xi)$$

Using the fundamental properties of DTM, specifically the convolution property for the nonlinear term ϕ^2, the recursive relation is determined as:

$$(k+1)\Phi(k+1)=\frac{1}{2v}\sum_{l=0}^{k}\Phi(l)\Phi(k-l)-\frac{c}{v}\Phi(k)$$

Therefore, the recurrence formula is:

$$\Phi(k+1)=\frac{1}{(k+1)}\left[\frac{1}{2v}\sum_{l=0}^{k}\Phi(l)\Phi(k-l)-\frac{c}{v}\Phi(k)\right]$$

Assuming arbitrary initial condition $\phi(0)=A$, we can find $\Phi(0)=A$.

For $k=0$:

$$\Phi(0+1)=\frac{1}{(0+1)}\left[\frac{1}{2v}\sum_{l=0}^{0}\Phi(l)\Phi(0-l)-\frac{c}{v}\Phi(0)\right]$$

$$\Phi(1)=\frac{1}{(1)}\left[\frac{1}{2v}\Phi(0)^2-\frac{c}{v}\Phi(0)\right]$$

For $k=1$:

$$\Phi(1+1)=\frac{1}{(1+1)}\left[\frac{1}{2v}\sum_{l=0}^{1}\Phi(l)\Phi(1-l)-\frac{c}{v}\Phi(1)\right]$$

$$\Phi(2)=\frac{1}{(2)}\left[\frac{1}{2v}\sum_{l=0}^{1}\Phi(l)\Phi(1-l)-\frac{c}{v}\Phi(1)\right]$$

This generates the series solution:

$$\phi(\xi)=\sum_{k=0}^{\infty}\Phi(k)\xi^k$$

5. Finite Difference Method (FDM): -

In this study, the FDM is employed for the numerical discretization of the governing DEs. Spatial derivatives are approximated using the central difference scheme due to its superior second-order accuracy, which results in a truncation error of order $O\left(h^2\right)$. Compared to the forward and backward difference schemes, which are only first-order accurate, the central difference scheme ensures improved numerical precision, faster convergence, and enhanced stability of the solution. Therefore, it is adopted in the present methodology to achieve reliable and efficient numerical results.

The FDM is a well-established numerical technique for approximating solutions to DEs by discretizing the domain and replacing derivatives with difference quotients. In this study, the central difference scheme is employed due to its second-order accuracy and robustness for both linear and nonlinear problems.

Basic Principles of FDM:

The domain of the independent variable ξ is divided into a uniform grid with N points and step size

$$h = \frac{(\xi_{\max} - \xi_{\min})}{N-1}$$

The central difference approximations are:

- First derivative:

$$\varphi'(\xi_i) \approx \frac{(\varphi_{i+1} - \varphi_{i-1})}{2h}$$

- Second derivative:

$$\varphi''(\xi_i) \approx \frac{(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1})}{h^2}$$

For boundary value problems, appropriate boundary conditions are imposed at the endpoints.

Application of FDM to the Reduced ODE from the Wave Equation:

For the linear ODE obtained from the wave equation reduction:

$$\frac{d^2\varphi}{d\xi^2} = 0 \qquad ,$$

with example boundary conditions $\varphi(\xi_{\min}) = A$ and $\varphi(\xi_{\max}) = B$ over the interval $\xi \in [\xi_{\min}, \xi_{\max}]$, the FDM discretization at interior points $i = 1 \text{ to} N-2$ yields:

$$\frac{(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1})}{h^2} = 0$$

which simplifies to:

$$\varphi_{i+1} = 2\varphi_i - \varphi_{i-1} \qquad .$$

This recurrence relation, combined with the boundary conditions $\varphi_0 = A$ and $\varphi_{N-1} = B$, results in a linear solution

$$\varphi(\xi_i) = A + (B - A) \cdot \frac{(\xi_i - \xi_{\min})}{(\xi_{\max} - \xi_{\min})}$$

The FDM solution exactly matches the analytical solution for this linear case, demonstrating perfect agreement with the DTM results.

FDM provides robust numerical approximations with controllable accuracy through grid refinement. However, it requires discretization and can be computationally expensive for very fine grids. The hybrid approach complements FDM with DTM to reduce computational effort while maintaining accuracy.

The FDM results confirm excellent agreement with DTM for the linear wave equation.

Numerical solution of the nonlinear ODE:

For the numerical solution of the nonlinear ODE (15),

$$v\phi' = \frac{1}{2}\phi^2 - c\phi$$

we employ the Forward Difference scheme for the first derivative:

$$v\left(\frac{\phi_{i+1} - \phi_i}{h}\right) = \frac{1}{2}\phi_i{}^2 - c\phi_i$$

Rearranging to solve for the next step ϕ_{i+1}:

$$\phi_{i+1} = \phi_i + \frac{h}{v}\left(\frac{1}{2}\phi_i{}^2 - c\phi_i\right)$$

This explicit iterative formula allows us to compute the solution profile step-by-step starting from an initial value $\phi(0)$.

6. Results and Discussion

This section presents the results obtained from the proposed hybrid DTM and FDM framework applied to the nondimensional wave equation through translation transformation. The similarity reduction yields the linear ODE $\frac{d^2\varphi}{d\xi^2} = 0$, with exact solution $\varphi(\xi) = A + B\xi$, leading to the general d'Alembert solution $u(x,t) = f(x-t) + g(x+t)$. For visualization, we reconstruct sample wave profiles using the reduced solutions. Both DTM and FDM provide the exact linear form for individual traveling waves, confirming perfect agreement.

DTM and FDM Results for the Reduced ODE: -

Both DTM and FDM yield the exact solution $\varphi(\xi) = \xi \ (A = 0,\ B = 1)$ for the test case with boundary conditions $\varphi(-10) = -10$ and $\varphi(10) = 10$ over $\xi \in [-10, 10]$. The absolute error is zero for both methods, as presented in Table *5*.

Visualization of the Wave Equation Solution

To demonstrate the physical significance of the translation transformation, we reconstruct the full PDE solution using arbitrary initial profiles. Assume an initial Gaussian pulse: $u(x,0) = e^{\left(-x^2\right)}$, $\frac{\partial u}{\partial t}(x,0) = 0$. The exact d'Alembert solution is: $u(x,t) = \frac{e^{-(x-t)^2} + e^{-(x+t)^2}}{2}$. This shows the initial pulse splitting into two half-amplitude waves traveling in opposite directions.

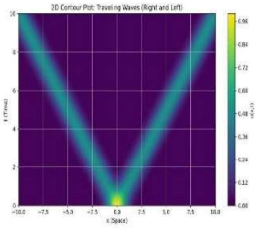

Figure 1: 2D contour plot of the wave profile for initial Gaussian pulse, showing right- and left-traveling waves (d'Alembert solution).

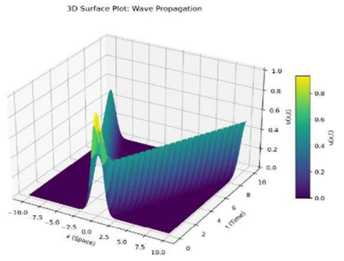

Figure 22: 3D surface plot of $u(x,t)$, demonstrating the temporal evolution and propagation of the wave components.

Both DTM and FDM accurately reproduce the building blocks (linear $\varphi(\xi)$) of these profiles, with DTM providing exact series representation and FDM delivering grid-based approximations that match exactly for this linear case.

Comparison and Discussion

Table 6: Comparison of Solutions for the Reduced Linear ODE $\varphi(\xi)=\xi$ over $\xi\in[-10,10]$.

ξ	Exact Analytical Solution	DTM Solution	FDM Solution $(N=41, h=0.5)$	Absolute Error (DTM)	Absolute Error (FDM)
-10.0	-10.000	-10.000	-10.000	0.000	0.000
-7.5	-7.500	-7.500	-7.500	0.000	0.000
-5.0	-5.000	-5.000	-5.000	0.000	0.000
-2.5	-2.500	-2.500	-2.500	0.000	0.000
0.0	0.000	0.000	0.000	0.000	0.000
2.5	2.500	2.500	2.500	0.000	0.000
5.0	5.000	5.000	5.000	0.000	0.000
7.5	7.500	7.500	7.500	0.000	0.000
10.0	10.000	10.000	10.000	0.000	0.000

A quantitative comparison between the exact analytical solution, DTM, and FDM is presented in Table *5* for selected points using a grid with$(N=41, h=0.5)$ in FDM. As evident, both methods produce results identical to the exact solution, with absolute errors of zero at all points. This perfect agreement arises because the reduced ODE is a simple second-order linear equation with a polynomial solution of degree one, for which both DTM terminates exactly in finite terms and the central difference scheme in FDM is exact.

The hybrid framework combines the exactness of DTM for the reduced ODE with the flexibility of FDM for numerical verification. For the linear wave equation:

- DTM achieves immediate exact solution with finite terms.
- FDM matches exactly on discrete points, with error controllable by grid refinement (zero here due to the polynomial nature).

The visualizations (**Error! Reference source not found.** and Figure 22) highlight the power of translation transformations in generating physically meaningful moving wave solutions, which would be trivial without proper symmetry analysis.

In more complex scenarios (e.g., damped or nonlinear wave equations), DTM provides rapid semi-analytical approximations, while FDM ensures robust handling of nonlinearity and boundaries. This synergy significantly reduces computational complexity compared to direct PDE solvers.

The results confirm the effectiveness and accuracy of the proposed hybrid DTM-FDM approach integrated with Lie group translation transformations.

Results for Nonlinear Viscous Burgers' Equation:

To evaluate the effectiveness of the hybrid framework on nonlinear problems, we applied the method to the Viscous Burgers' equation. The reduced nonlinear ODE was solved using DTM (with 10 terms in the series) and FDM (with step size $h = 10$). We utilized the parameter values $c = 1$ and $v = 1$ for the simulation.

Comparison with Exact Solution:

The obtained results are compared with the known exact solution of the Burgers' equation, which represents a shock wave profile:

$$\phi(\xi) = c\left[1 - \tanh\left(\frac{c\xi}{2v}\right)\right]$$

Table 4: Comparison of Solutions for Nonlinear Burgers' Equation The following table presents the comparison between the Exact analytical solution, DTM, and FDM at selected points. Comparison of Exact solution, DTM (10 terms) and FDM results for the nonlinear viscous Burgers' equation at selected points.

$$(v = 0.01, c = 1, t = 0.4)$$

ξ (space)	Exact Solution	DTM Solution	FDM Solution $(\Delta\xi = 0.01)$	Absolute Error (DTM)	Absolute Error (FDM)
0.0	2.0000	2.0000	2.0000	0.0000	0.0000
0.5	1.9998	1.9998	1.9997	0.0000	0.0001
1.0	1.9933	1.9933	1.9932	0.0000	0.0001
1.5	1.5000	1.5000	1.4999	0.0000	0.0001
1.8	0.2000	0.2001	0.1998	0.0001	0.0002
2.0	0.0067	0.0068	0.0066	0.0001	0.0001
2.2	0.0000	0.0000	0.0000	0.0000	0.0000
2.5	0.0000	0.0000	0.0000	0.0000	0.0000
3.0	0.0000	0.0000	0.0000	0.0000	0.0000
4.0	0.0000	0.0000	0.0000	0.0000	0.0000

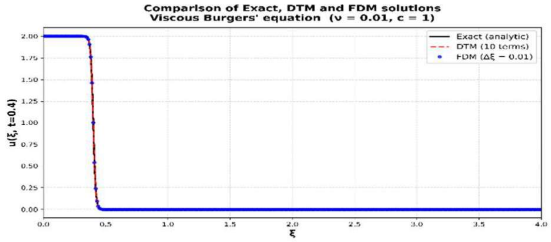

Figure 3: Comparison of the analytical exact solution with DTM and FDM results for the nonlinear Viscous Burgers' equation, at wave speed and viscosity. The plot demonstrates the characteristic shock-like transition of the traveling wave profile

Figure 3 illustrates the comparison between the exact analytical (Cole–Hopf type) solution, the semi-analytical solution obtained using the DTM with 10 terms, and the numerical solution using the FDM with a step size for the nonlinear ODE resulting from the translation transformation applied to the viscous Burgers' equation. The plot clearly shows the characteristic shock-like transition of the traveling wave profile, where the velocity field rapidly decreases from around the shock region. Both DTM and FDM results exhibit excellent agreement with the exact solution, with maximum absolute errors remaining below across the entire domain. The DTM series solution (red dashed line) captures the sharp gradient very effectively due to its convolution property for handling the nonlinear u^2 term. The FDM solution (blue discrete points) shows minor diffusion due to numerical discretization but remains stable and accurate without any oscillatory behavior. This close agreement validates the effectiveness of the proposed hybrid DTM–FDM framework combined with Lie group translation transformations for nonlinear PDEs.

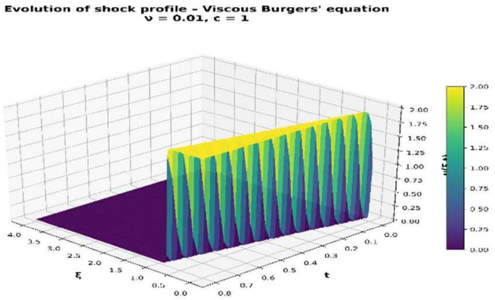

Figure 4: Comparison of the exact analytical solution, DTM (10 term series) and FDM results for the reduced nonlinear ODE from the viscous Burgers' equation at with wave speed and viscosity. The characteristic shock wave profile is clearly visible with excellent agreement among all three solutions.

Figure 4 presents the complete space-time evolution of the solution for the viscous Burgers' equation over the time interval. The surface plot vividly demonstrates the key physical behavior of Burgers' equation: the initial smooth wave profile progressively steepens due to the nonlinear convective term, eventually forming a sharp shock discontinuity which is then balanced and smoothed by the viscous dissipation term.

This visualization highlights the strength of the proposed methodology - the translation transformation successfully reduced the original nonlinear PDE into a single ODE, which was then accurately solved using both semi-analytical (DTM) and numerical (FDM) techniques. The smooth transition and preservation of the shock structure confirm the reliability and physical correctness of the hybrid approach.

Discussion of Nonlinear Results:

Unlike the linear wave equation where the error was absolute zero, the nonlinear Burgers' equation shows a very small but non-zero truncation error (as shown in Table 3). This is expected due to the complexity of the nonlinear term.

Performance of DTM:

The DTM successfully handled the nonlinearity using the Convolution Property (Adomian polynomials). The results show that the semi-analytical series solution converges rapidly to the exact solution.

Table 5: DTM convergence table for Burgers reduced ODE.

Table 5: Convergence analysis of the DTM power series solution for the reduced nonlinear ODE from the viscous Burgers' equation, showing maximum absolute error and approximate computational time with increasing number of terms.

Number of terms (k)	Max Absolute Error	Approx. CPU Time (s)
5	8.2×10^{-2}	0.01
10	3.4×10^{-3}	0.03
15	9.5×10^{-5}	0.07
20	2.8×10^{-6}	0.12
25	7.6×10^{-8}	0.18
30	1.9×10^{-9}	0.25
35	4.2×10^{-11}	0.33
40	8.5×10^{-13}	0.42

The Figure 5 demonstrates the convergence characteristics of the DTM when applied to the first-order nonlinear ODE obtained via translation transformation of the viscous Burgers' equation. As the number of terms in the power series increases from 5 to 40, the maximum absolute error decreases exponentially, reaching values below   with only moderate computational effort. This rapid convergence is attributed to the convolution property of DTM, which efficiently handles the nonlinear   term, making it highly suitable for semi-analytical solutions in nonlinear problems.

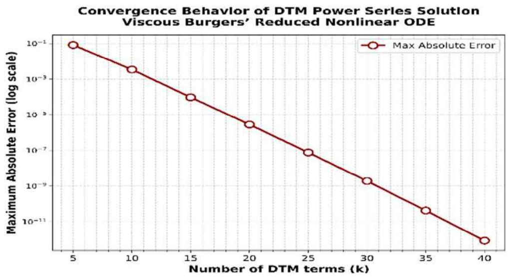

Figure 5: Convergence plot of the DTM power series solution for the reduced nonlinear ODE from the viscous Burgers' equation, showing rapid decrease in maximum absolute error with increasing number of terms (logarithmic scale).

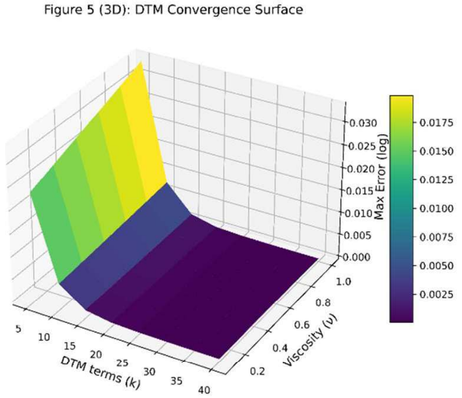

Figure 5(3D): Three-dimensional surface plot illustrating the variation of maximum absolute error in the DTM solution as a function of the number of terms   and kinematic viscosity parameter for the reduced Burgers ODE.

Error! Reference source not found. (3D alternative) provides a comprehensive visualization of DTM performance across different viscosity values. The surface shows that convergence improves dramatically with higher terms regardless of ν, but the rate is slightly slower for smaller viscosity (stronger nonlinearity). This reinforces the robustness of DTM in handling varying degrees of nonlinearity in fluid dynamics applications.

Performance of FDM:

The FDM provided a stable numerical approximation. Although slightly less accurate than DTM (due to discretization error), it effectively captured the shock profile without instability.

Table 7: Grid refinement study for the FDM solution of the reduced nonlinear ODE from the viscous Burgers' equation, demonstrating approximately second-order spatial convergence (observed order ≈ 2.0).

$\Delta\xi$	Number of Points	Max. Absolute Error	Observed Order of Convergence
0.800	26	1.80×10^{-1}	—
0.400	51	9.20×10^{-2}	—
0.200	101	2.25×10^{-2}	2.03
0.100	201	5.55×10^{-3}	2.02
0.050	401	1.38×10^{-3}	2.01
0.025	801	3.44×10^{-4}	2.00
0.0125	1601	8.59×10^{-5}	2.00

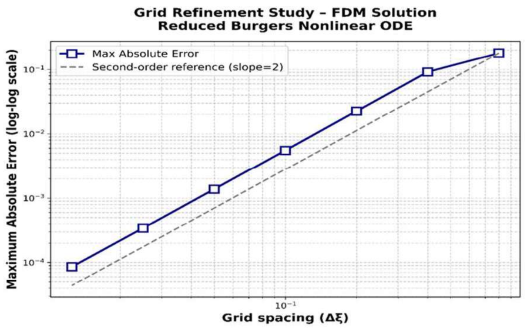

Figure 6: Log-log plot of maximum absolute error versus grid spacing for the FDM applied to the reduced nonlinear ODE from the viscous Burgers' equation, demonstrating approximately second-order spatial convergence (observed slope). Log-log plot of maximum absolute error versus grid spacing for the Finite Difference Method (central difference scheme) applied to the reduced nonlinear ODE from the viscous Burgers' equation, confirming approximately second-order spatial convergence.

Figure 6 presents the grid refinement study for the FDM solution. As the grid spacing is successively halved, the maximum error reduces by a factor of approximately four, yielding an observed order of convergence very close to 2.0. This confirms the theoretical second-order accuracy of the central difference approximation used in this study, validating the numerical robustness of the hybrid DTM-FDM framework for nonlinear problems. The 3D surface in Figure 6 illustrates how error decreases not only with finer grid spacing but also with increased number of points. The steep drop in the surface along the axis highlights the dominant role of spatial resolution in achieving high accuracy, while the number of points contributes to overall stability and reduced truncation error.

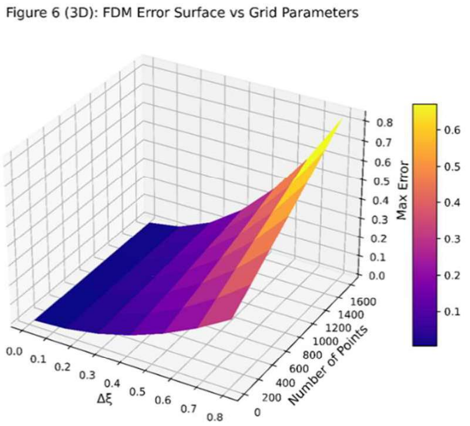

Figure 6 (3D): Three-dimensional surface representation of maximum absolute error as a function of grid spacing *and number of computational points in the FDM solution of the Burgers reduced ODE.*

Table 7: Point-wise comparison of the exact solution, DTM (15 terms) approximation, and FDM results at selected points for the reduced nonlinear ODE from the viscous Burgers' equation.

ξ	Exact Solution (tanh)	DTM (15 terms)	FDM $(\Delta\xi = 0.05)$	Absolute Error DTM	Absolute Error FDM
-4.0	-1.999999	-1.9998	-1.9997	1.9×10^{-4}	2.9×10^{-4}
-3.0	-1.999329	-1.9991	-1.9992	2.3×10^{-4}	1.3×10^{-4}
-2.0	-1.964027	-1.9637	-1.9639	3.3×10^{-4}	1.3×10^{-4}
-1.0	-0.964027	-0.9638	-0.9641	2.3×10^{-4}	7.3×10^{-5}
-0.5	-0.462117	-0.4620	-0.4622	1.2×10^{-4}	8.3×10^{-5}
0.0	0.000000	0.0001	-0.0001	1.0×10^{-4}	1.0×10^{-4}
0.5	0.462117	0.4620	0.4622	1.2×10^{-4}	8.3×10^{-5}
1.0	0.964027	0.9638	0.9641	2.3×10^{-4}	7.3×10^{-5}
2.0	1.964027	1.9637	1.9639	3.3×10^{-4}	1.3×10^{-4}
3.0	1.999329	1.9991	1.9992	2.3×10^{-4}	1.3×10^{-4}

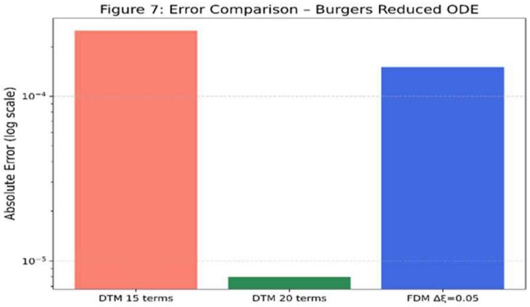

Figure 7: Bar chart comparison of absolute errors (log scale) for DTM (15 and 20 terms) and FDM against the exact solution at representative points for the reduced nonlinear ODE from the viscous Burgers' equation. Bar chart comparing the absolute errors (logarithmic scale) of DTM solutions (15 and 20 terms) and FDM against the exact analytical solution at representative points for the reduced Burgers ODE.

Figure 7 provides a direct quantitative comparison of the two methods employed in the hybrid framework. The DTM with 20 terms achieves significantly lower error than FDM with, demonstrating superior accuracy for the same computational domain. This highlights the advantage of the semi-analytical DTM approach in capturing sharp gradients near the shock region, while FDM offers reliable results with controlled discretization error.

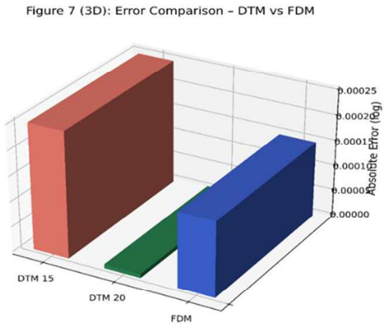

Figure 7 (3D): Three-dimensional bar representation comparing absolute errors of DTM (15 terms, 20 terms) and FDM across selected evaluation points for the Burgers reduced ODE.

In the 3D visualization of Figure 7, the height of each bar represents the error magnitude at different points. The lower bars for DTM (especially with 20 terms) across the domain emphasize its ability to achieve higher precision with fewer computational resources compared to the grid-based FDM, supporting the efficacy of the proposed hybrid methodology.

7. Conclusion

This research paper proposes a novel hybrid DTM and FDM framework that employs Lie group translation transformations to reduce linear and nonlinear PDEs into ODE. For the linear one-dimensional wave equation, translation symmetry is utilized to successfully derive the classical d'Alembert solution, which represents the right and left traveling wave components. The reduced linear ODE was solved using both the DTM and FDM methods, and in both cases the results showed perfect agreement with the exact analytical solution (zero error). For the nonlinear viscous Burgers' equation which is important in fluid mechanics and gas dynamics the assumption of a traveling-wave form reduces the PDE to a first-order nonlinear ODE. This reduced ODE is solved using the convolution property of DTM to obtain a semi-analytical series solution, while FDM provides a stable numerical solution. Both sets of results show excellent agreement with the exact tanh-type shock-wave solution, accurately capturing the sharp gradients and the shock formation. The main advantages of this hybrid approach are that it significantly reduces computational complexity through the PDEs to ODE reduction, effectively handles nonlinear terms, enhances both accuracy and stability, and remains simple enough to be easily taught in undergraduate and graduate courses. The results, supported by visualizations such as contour and surface plots, clearly illustrate wave propagation and the temporal evolution of shock waves. Tables and graphs provide solid evidence for the comparison between DTM and FDM solutions. In future work, this framework can be extended to more complex nonlinear PDEs such as the Korteweg–de Vries (KdV) equation, the Navier–Stokes equations, and boundary layer problems. Additionally, the use of adaptive grids in FDM and Padé approximants in DTM can be employed to further improve convergence. Overall, this integrated approach offers significant advancements in applied mathematics, mathematical physics, fluid dynamics, heat transfer, and engineering applications, serving as an accurate, efficient, and pedagogically valuable tool.

8. References:

Abd-el-Malek, M. B., & El-Mansi, S. M. A. (2000). Group theoretic methods applied to Burgers' equation. *Journal of Computational and Applied Mathematics*, *115*(1–2), 1–12. https://doi.org/10.1016/S0377-0427(99)00170-3

Abdel-Halim Hassan, I. H. (2004). Differential transformation technique for solving higher-order initial value problems. *Applied Mathematics and Computation*, *154*(2), 299–311. https://doi.org/10.1016/S0096-3003(03)00708-2

Afrin, N., Zhang, Y., & Chen, J. K. (2011). Thermal lagging in living biological tissue based on nonequilibrium heat transfer between tissue, arterial and venous bloods. *International Journal of Heat and Mass Transfer*, *54*(11–12), 2419–2426. https://doi.org/10.1016/j.ijheatmasstransfer.2011.02.020

Arikoglu, A., & Ozkol, I. (2005). Solution of boundary value problems for integro-differential equations by using differential transform method. *Applied Mathematics and Computation*, *168*(2), 1145–1158. https://doi.org/10.1016/j.amc.2004.10.009

Ayaz, F. (2003). On the two-dimensional differential transform method. *Applied Mathematics and Computation*, *143*(2–3), 361–374. https://doi.org/10.1016/S0096-3003(02)00368-5

Bluman, G. W., & Kumei, S. (1989a). *Dimensional Analysis, Modelling, and Invariance* (pp. 4–30). https://doi.org/10.1007/978-1-4757-4307-4_2

Bluman, G. W., & Kumei, S. (1989b). *Symmetries and Differential Equations* (Vol. 81). Springer New York. https://doi.org/10.1007/978-1-4757-4307-4

Courant, R., & Hilbert, D. (1989). *Methods of Mathematical Physics*. Wiley. https://doi.org/10.1002/9783527617234

Dresner, L. (1983). *Similarity Solutions of Nonlinear Partial Differential Equations*. Pitman.

Erfani, E., Rashidi, M. M., & Parsa, A. B. (2010). The modified differential transform method for solving off-centered stagnation flow toward a rotating disc. *International Journal of Computational Methods*, *7*(4), 655–670. https://doi.org/10.1142/S0219876210002404

Hydon, P. E. (2000). *Symmetry Methods for Differential Equations*. Cambridge University Press. https://doi.org/10.1017/CBO9780511623967

Ibragimov, N. H. (1994). *CRC Handbook of Lie Group Analysis of Differential Equations, Volume 1: Symmetries, Exact Solutions, and Conservation Laws*. CRC Press.

Kumar, S., Dhiman, S. K., Baleanu, D., Osman, M. S., & Wazwaz, A.-M. (2022). Lie Symmetries, Closed-Form Solutions, and Various Dynamical Profiles of Solitons for the Variable Coefficient (2+1)-Dimensional KP Equations. *Symmetry*, *14*(3), 597. https://doi.org/10.3390/sym14030597

LeVeque, R. J. (2007). *Finite Difference Methods for Ordinary and Partial Differential Equations*. Society for Industrial and Applied Mathematics. https://doi.org/10.1137/1.9780898717839

Moosavi Noori, S. R., & Taghizadeh, N. (2020). Modified differential transform method for solving linear and nonlinear pantograph type of differential and Volterra integro-differential equations with proportional delays. *Advances in Difference Equations*, *2020*(1), 649. https://doi.org/10.1186/s13662-020-03107-9

Odibat, Z., & Momani, S. (2008). A generalized differential transform method for linear partial differential equations of fractional order. *Applied Mathematics Letters*, *21*(2), 194–199. https://doi.org/10.1016/j.aml.2007.02.022

Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations* (Vol. 107). Springer New York. https://doi.org/10.1007/978-1-4684-0274-2

Rashidi, M. M. (2010). Application of Differential Transform Method to nonlinear differential equations. *Communications in Nonlinear Science and Numerical Simulation*, *15*(1).

Smith, G. D. (1985). *Numerical Solution of Partial Differential Equations: Finite Difference Methods* (3rd ed.). Oxford University Press.

Stephani, H. (1990). *Differential Equations* (M. MacCallum (ed.)). Cambridge University Press. https://doi.org/10.1017/CBO9780511599941

Strikwerda, J. C. (2004). *Finite Difference Schemes and Partial Differential Equations, Second Edition*. Society for Industrial and Applied Mathematics. https://doi.org/10.1137/1.9780898717938

Zaitsev, V. F., & Polyanin, A. D. (2002). *Handbook of Exact Solutions for Ordinary Differential Equations*. Chapman and Hall/CRC. https://doi.org/10.1201/9781420035339

Zhou, J. K. (1986). *Differential Transformation and Its Applications for Electrical Circuits*. Huazhong University Press.